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ON THE IDENTITIES OF SUBALGEBRAS OF MATRICES OVER THE GRASSMANN ALGEBRA

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ABSTRACT
The polynomial identities of certain subalgebras of matrices, over the Grassmann algebra, are studied in terms of their cocharacters. Our present knowledge of such characters for matrices over a field F (char $F = 0$) plays a role here, and some of these results are extended to these subalgebras. In particular, we obtain bounds for the codimensions of these algebras (Theorem 0.1 below).

§0. Introduction

The quantitative study of the set of identities of a given P.I. algebra (in characteristic zero) is done by studying its codimensions and its cocharacters or, equivalently its Poincaré series. We shall assume the reader has some familiarity with the representation theory of the symmetric group S_n, and with cocharacters and codimensions of P.I. algebras.

The present work should be viewed as a first step towards calculating these series for certain K-semiprime algebras which were introduced by Kemer [7]. The K-prime algebras are either $M_k(F) = F_k$, $M_k(E) = E_k$ (E is the Grassmann algebra) or certain subalgebras $E_{k,l} \subseteq E_{k+l}$. These algebras play an important role in the theory of P.I. algebras. One consequence of our work here is the following:

0.1. THEOREM. *Let A be one of the above algebras and let $\{c_n(A)\}_{n=1}^{\infty}$ be its codimension series. Then there exist (explicit) constants a, c_1, c_2, g_1 and g_2 such that for all n,*

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$$c_1 \cdot \left(\frac{1}{n}\right)^{g_1} \cdot a^n \leqq c_n(A) \leqq c_2 \cdot \left(\frac{1}{n}\right)^{g_2} \cdot a^n.$$

In particular, the exponential growth of the codimensions $\{c_n(A)\}$ of such algebras is (explicitly) captured. The work of Kemer [7] highly motivates the study of these algebras. Following Gateva [6] we first summarize, in §1, Kemer's results.

One of these algebras is F_k, which is clearly of central importance, and an extensive study has been made towards understanding its polynomial identities. We therefore briefly summarize now some of the results about the cocharacters of F_k:

Let $\chi_n(A)$ be the (n-th) cocharacter of the algebra A, and write

$$\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda(A) \cdot \chi_\lambda$$

($m_\lambda(A) \in \mathbb{N}$ are the multiplicities of the irreducible S_n characters χ_λ in $\chi_n(A)$). It follows from [11] that

$$\chi_n(F_k) = \sum_{\substack{\lambda \in \mathrm{Par}(n) \\ h(\lambda) \leq k^2}} m_\lambda(F_k) \cdot \chi_\lambda.$$

($h(\lambda) \leqq r$ if $(\lambda_1, \lambda_2, \ldots)$ and $\lambda_{r+1} = 0$.)

Write now

$$\sum_{\substack{\mu \in \mathrm{Par}(n) \\ h(\mu) \leq k}} \chi_\mu \otimes \chi_\mu = \sum_{\substack{\lambda \in \mathrm{Par}(n) \\ h(\lambda) \leq k^2}} \bar{m}_\lambda \cdot \chi_\lambda$$

($\otimes$ is the Kronecker product). It follows from [4], [5] that if $\lambda_{k^2} \geqq 2$ then

0.2. THEOREM. $\bar{m}_\lambda = m_\lambda(F_k)$.

Thus, the problem of calculating $\chi_n(F_k)$ is essentially equivalent to that of calculating

$$\sum_{\substack{\mu \in \mathrm{Par}(n) \\ h(\mu) \leq k}} \chi_\mu \otimes \chi_\mu,$$

a problem which is still open (except for $k = 2$) and seems to be very hard.

Nevertheless, these results imply.

0.3. THEOREM [15]. *Let $a_n \simeq b_n$ indicate that*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

Then

$$c_n(F_k) \simeq c \cdot \left(\frac{1}{n}\right)^g \cdot k^{2n},$$

where

$$c = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \left(\frac{1}{2}\right)^{(k^2-1)/2} \cdot 1! \cdots (k-1)! \cdot k^{(k^2+4)/2}$$

and

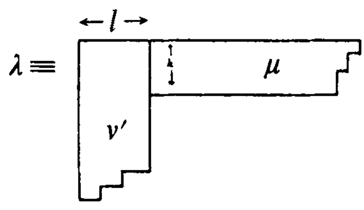
$$g = (k^2 - 1)/2.$$

Thus Theorem 0.1 is an extension of the above theorem.

We now describe the main results of this paper. Identify a partition λ with its Young diagram, and let

$$\lambda = (\lambda_1, \lambda_2, \ldots) \in H(k, l; n) \qquad (\text{i.e., } \lambda_{k+1} \leqq l)$$

such that $\lambda_k \geqq l$. Then



where $h(\mu) \leqq k$ and $h(\nu) \leqq l$ (see [2]), and we denote $\lambda \mapsto (\mu, \nu')$.

It follows from [9], [3] that

$$\chi_n(E_k) = \sum_{\lambda \in H(k^2, k^2; n)} m_\lambda(E_k) \cdot \chi_\lambda.$$

0.4. Theorem. *Our main result reads as follows:*

Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in H(k^2, k^2; n)$, $\lambda_{k^2} \geqq k^2$ and denote $\lambda \mapsto (\mu, \nu')$. If μ and ν are large enough, then $m_\lambda(E_k) \geqq 1$. [We do believe that $m_\lambda(E_k)$ is close to $m_\mu(F_k) \cdot m_\nu(F_k)$.]

The proof requires a considerable amount of calculations with various idempotents in FS_n, viewed as polynomials, and with substitutions in E_k. The right choices of idempotents and substitutions yield the proof.

The inequality $m_\lambda(E_k) \geqq 1$ and the asymptotics of the degrees $d_\lambda = \deg(\chi_\lambda)$

imply the lower bound

$$c_1\left(\frac{1}{n}\right)^{g_1}(2k^2)^n \leqq c_n(E_k)$$

with c_1, g_1 explicit. Since $E_k = F_k \otimes E$,

$$c_n(E_k) \leqq c_n(F_k)\cdot c_n(E),$$

and a similar upper bound for $c_n(E_k)$ follows from the (asymptotically) known series $c_n(F_k)$ and $c_n(E)$.

This proves Theorem 0.1 for the algebras E_k.

The results of [1], [2], [3] and [7] allow us to extend Theorem 0.1 to the algebras $E_{k,l}$. Finally, it is not difficult to extend these results to any K-semiprime algebra.

§1. Kemer's results [7]

Let $A_k = M_k(A)$ denote the $k \times k$ matrices over the algebra A (any A). Let $E = E(V)$ be the Grassmann (Exterior) algebra of a countable dimension vector space V over a field F, char $F = 0$. By considering the length of the basis elements of E we have that

$$E = E_0 \oplus E_1,$$

where E_0 (resp. E_1) is spanned by the elements of even (resp. odd) length. Given $k, l \geqq 0$, we denote by $E_{k,l} = M_{k,l}(E)$ the following subalgebra of E_{k+l}:

$$E_{k,l} = \left\{\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right) \,\middle|\, A \in M_k(E_0),\ D \in M_l(E_0),\ B \text{ is } k \times l \text{ and } C \text{ is } l \times k,\ \text{both with entries in } E_1.\right\}$$

We consider now algebras with 1. Kemer defines the property of K-semiprimeness as follows:

K-ideals (Kemer calls them T-ideals) are obtained from T ideals by taking all possible evaluations.

The relatively free algebra in a given variety is called K-semiprime if it does not contain nilpotent K-ideals, and in that case, the variety itself is called K-semiprime. K-primeness is defined in an analogous way and a K-semiprime algebra is a finite direct sum of K-prime algebras. A. Berele showed me that K-primeness is equivalent to the following property:

K-primeness: The algebra A is K-prime if it satisfies the following property: Let $f(x_1, \dots, x_r)$, $g(x_1, \dots, x_s)$ be polynomials such that

$$f(x_0, \dots, x_{r-1})x_r g(x_{r+1}, \dots, x_{r+s})$$

is an identity for A, then either f or g is an identity for A. Equivalently, an algebra A is K-prime if no product of non-zero K-ideals of A is zero.

In the work of Kemer [7], the following two theorems are relevant to this paper:

1.1. THEOREM [Kemer]. *Any K-prime variety is generated by one of the following algebras*: $F_k = M_k(F)$; $E_k = M_k(E) = F_k \otimes E$; $E_{k,l} = M_{k,l}(E)$ *where* $l \leqq k$.

Let A, B be two P.I. algebras. Denote $A \sim B$ if they satisfy the same set of identities.

1.2. THEOREM [Kemer]. *The next equivalences hold*:

$$E_{1,1} \sim E \otimes E, \tag{1.2.1}$$

$$E_{k,l} \otimes E \sim E_{k+l}, \tag{1.2.2}$$

$$E_{k,l} \otimes E_{p,q} \sim E_{kq+lp,kp+lq}. \tag{1.2.3}$$

The importance of these algebras lies in the following:

1.3. THEOREM [Kemer]. *Every relatively free algebra A has a maximal nilpotent K-ideal I such that A/I is K-semiprime.*

§2. Some preliminaries

2.1. The partitions $\lambda \vdash m$ decompose FS_m:

$$FS_m = \bigoplus_{\lambda \vdash m} I_\lambda,$$

I_λ minimal two-sided ideals.

A Young tableau T_λ of shape λ defines the two subgroups $R_{T_\lambda}, C_{T_\lambda} \subseteq S_m$:

R_{T_λ} = the row permutations of T_λ,

R_{T_λ} = the column permutations of T_λ.

Denote

$$\bar{R}_{T_\lambda} = \sum_{p \in R_{T_\lambda}} p, \qquad \bar{C}_{T_\lambda} = \sum_{q \in C_{T_\lambda}} \operatorname{sgn}(q) \cdot q.$$

Usually, one constructs now, in FS_m, the primitive idempotent

$$e_{T_\lambda} = \alpha \cdot \bar{R}_{T_\lambda} \cdot \bar{C}_{T_\lambda},$$

where α^{-1} is the product of the hook numbers of λ. Such idempotents, that correspond to the standard tableaux of shape λ, completely decompose I_λ:

$$I_\lambda = \bigoplus_{T_\lambda \text{ standard}} (FS_m) \cdot e_{T_\lambda}.$$

Note that one can also construct $f_{T_\lambda} = \alpha \cdot \bar{C}_{T_\lambda} \cdot \bar{R}_{T_\lambda}$; these again are primitive idempotents with the same property as the e_{T_λ}'s.

2.2. Given $a = \Sigma_{\sigma \in S_m} \alpha_\sigma \sigma \in FS_m$ ($\alpha_\sigma \in F$), we identify a with the polynomial

$$a \equiv a(x_1, \dots, x_m) = \sum_{\sigma \in S_m} \alpha_\sigma M_\sigma(x_1, \dots, x_m),$$

where $M_\sigma(x_1, \dots, x_m) = x_{\sigma(1)} \cdots x_{\sigma(m)}$. This applies, in particular, to the idempotents $e_{T_\lambda} \equiv e_{T_\lambda}(x_1, \dots, x_m)$.

2.3. $\chi_n(A) = \Sigma_{\lambda \vdash n} m_\lambda(A) \chi_\lambda$ is the cocharacter of the P.I. algebra A with multiplicities $m_\lambda(A)$; χ_λ is the irreducible S_n character that corresponds to λ.

2.4. If $m_\mu(A) \neq 0$ then, for some tableau T_μ,

$$e_{T_\mu}(x_1, \dots, x_n) \notin \mathrm{Id}(A) = Q,$$

and for some — possibly another — tableau $\tilde{T}_\mu$,

$$f_{\tilde{T}_\mu}(x_1, \dots, x_n) \notin Q,$$

$\mathrm{Id}(A) = Q$ being the identities of A.

2.5. The results of [4], [5], imply that for most μ's, $m_\mu(F_k) \neq 0$. In particular, let $n \geqq 2k^2$, $n = wk^2 + r$, $0 \leqq r < k^2$, and define

$$\mu = (w + r, \underbrace{w, w, \dots, w}_{k^2 - 1}) \vdash n.$$

It then follows from [5] and from [12, Th.1.3] that $m_\mu(F_k) \geqq 1$: for an appropriate choice of T_μ, $e_{T_\mu}(x_1, \dots, x_n)$ (or $f_{T_\mu}(x_1, \dots, x_n)$) is not an identity of F_k. Thus, for generic $k \times k$ matrices $X_1, \dots, X_n$, $e_{T\mu}(X_1, \dots, X_n) \neq 0$ ($f_{T_\mu}(X_1, \dots, X_n) \neq 0$).

2.6. Remark. Let $\nu \vdash n$, T_ν a tableau of shape ν, T'_ν the conjugate tableau (of shape ν', the conjugate partition). Clearly, $R_{T'_\nu} = C_{T_\nu}$ and $C_{T'_\nu} = R_{T_\nu}$.

2.7. Recall that $E = E(V)$ is the Grassmann algebra of a countable dimensional vector space V. Let $v_1, v_2, \ldots$ denote a basis of V.

LEMMA. *Let T_ν be a tableau, T'_ν its conjugate, with their corresponding e and f idempotents as in* 2.1. *Also, let* $V = \text{span}_F\{v_1, v_2, \ldots\}$, $E = E(V)$ *as above, and consider* $E \otimes_F F\langle x\rangle$. *Then*

$$e_{T'_\nu}(v_1 \otimes x_1, \ldots, v_n \otimes x_n) = (v_1 \cdots v_n) \otimes f_{T_\nu}(x_1, \ldots, x_n).$$

PROOF. Since $v_{\sigma(1)} \cdots v_{\sigma(n)} = \text{sgn}(\sigma) v_1 \cdots v_n$ $(v_i v_j = -v_j v_i)$, hence

$$M_\sigma(v_1 \otimes x_1, \ldots, v_n \otimes x_n) = v_1 \cdots v_n \otimes \text{sgn}(\sigma) M_\sigma(x_1, \ldots, x_n).$$

Thus

$$\begin{aligned} e_{T'_\nu}(v \otimes x) &= \sum_{\substack{\rho = q \in C_{T_\nu} \\ \gamma = p \in R_{T'_\nu}}} \text{sgn}(q) M_{pq}(v \otimes x) \\ &= (v_1 \cdots v_n) \otimes \sum_{\substack{\rho \in R_{T_\nu} \\ \gamma \in C_{T'_\nu}}} \text{sgn}(\rho) \cdot \text{sgn}(\gamma\rho) M_{\gamma\rho}(x) \\ &= (v_1 \cdots v_n) \otimes \sum_{\rho, \gamma} \text{sgn}(\gamma) M_{\gamma\rho}(x) \\ &= (v_1 \cdots v_n) \otimes f_{T'_\nu}(x). \end{aligned}$$

The Capelli identity for F_k will play an important role in what follows.

2.8. Recall that the Capelli polynomial $d_{n+1}[x; y]$ is defined as follows:

$$d_{n+1}[x; y] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_n x_{\sigma(n+1)}.$$

Recall also that $d_{k^2+1}[x; y]$ is an identity for F_k.

LEMMA. *Let R be any F-algebra,*

$$B_1, \ldots, B_{k^2} \in R_k = M_k(R) \qquad \text{and} \qquad D_1, \ldots, D_{k^2+1} \in F_k.$$

Then

$$d_{k^2+1}[D_1, \ldots, D_{k^2+1}; B_1, \ldots, B_{k^2}] = 0.$$

PROOF. Since d_{k^2+1} is multilinear, we may assume that $B_i = r_1 \otimes C_i$, $r_i \in R$, $C_i \in F_k$. The lemma now follows since

$$\begin{aligned} &d_{k^2+1}[D_1, \ldots, D_{k^2+1}; r_1 \otimes C_1, \ldots, r_{k^2} \otimes C_{k^2}] \\ &\quad = (r_1 \cdots r_{k^2}) \otimes d_{k^2+1}[D_1, \ldots, D_{k^2+1}; C_1, \ldots, C_{k^2}] = 0. \qquad \text{Q.E.D.} \end{aligned}$$

2.9. DEFINITION. Let $u_1, u_2, \ldots$ be basis elements of E (so either of even or of odd length; (§1)) and $\boldsymbol{u} = (u_1, \ldots, u_m)$, then define

$$\#\boldsymbol{u} = \operatorname{card}\{i \mid 1 \leqq i \leqq m,\ u_i \text{ is of even length}\}.$$

We shall consider substitutions in E_k of the form

$$u_i \to u_i \otimes D_i,$$

where $\boldsymbol{u} = (u_1, \ldots, u_m)$ as above, and $D_i \in F_k$.

2.10. LEMMA. *Let $h(x_1, \ldots, x_s)$ be a multilinear polynomial, let $k^2 + 1 \leqq l \leqq s$, $1 \leqq i_1 < \cdots < i_l \leqq s$ and assume h is alternating in $x_{i_1}, \ldots, x_{i_l}$. Substitute $x_i \to u_i \otimes D_i$ $1 \leqq i \leqq s$, as in 2.9. If $\#(u_{i_1}, \ldots, u_{i_l}) \geqq k^2 + 1$ then $h(u_1 \otimes D_1, \ldots, u_s \otimes D_s) = 0$.*

PROOF. W.L.O.G. $i_j = j$ and $u_1, \ldots, u_{k^2+1}$ are of even length; we can assume $u_1 = \cdots = u_{k^2+1} = 1$. Since $h(x)$ is in particular alternating in $x_1, \ldots, x_{k^2+1}$ we can write

$$h(x_1, \ldots, x_s) = \sum \alpha \cdot M_0 \cdot d_{k^2+1}[x_1, \ldots, x_{k^2+1}; M_1, \ldots, M_{k^2}]M_{k^2+1}$$

where in each summand, the M_i's are either equal to 1 or are monomials in some of the x_j's. Substituting $x_i \to u_i \otimes D_i$, the M_i's become $\bar{M}_i \in E_k$, while $x_i \to D_i \in F_k$ if $1 \leqq i \leqq k^2 + 1$. The proof now follows from Lemma 2.8.

Q.E.D.

2.11. Recall from 2.1 that for $\theta \vdash m$ and a tableau T_θ, we can write

$$\alpha^{-1} e_{T_\theta}(x) = \sum_{p \in R_{T_\theta}} p\bar{C}_{T_\theta}(x_1, \ldots, x_m) = \sum_{p \in R_{T_\theta}} \bar{C}_{T_\theta}(x_{p(1)}, \ldots, x_{p(m)}).$$

Correspond i with x_i, then the entries of the j-th column of T_θ correspond to a subset of $x_1, \ldots, x_m$, and $\bar{C}_T(x)$ is alternating in that subset. If ω is the number of columns of T_θ, then $\bar{C}_{T_\theta}(x)$ is a polynomial in ω subsets of variables, and is alternating in each such subset. Clearly, the same applies to $\bar{C}_{T_\theta}(x_{p(1)}, \ldots, x_{p(m)})$.

2.12. LEMMA. *Let $\theta \vdash m$ with T_θ a tableau with k^2 columns. Let $\boldsymbol{u} = (u_1, \ldots, u_m)$ as in 2.9, with $\#\boldsymbol{u} \gneqq k^4$, and let $X_1, \ldots, X_m$ be generic $k \times k$ matrices. Then*

$$e_{T_\theta}(u \otimes X) = e_{T_\theta}(u_1 \otimes X_1, \ldots, u_m \otimes X_m) = 0.$$

PROOF. By the above description of $\alpha^{-1} e_{T_\theta}(x)$ and since

$\#(u_{p(1)}, \ldots, u_{p(m)}) = \#(u_1, \ldots, u_m)$, it suffices to show that $\bar{C}_{T_\theta}(u \otimes X) = 0$.

Now, $\bar{C}_{T_\theta}(x)$ is alternating in each of its k^2 subsets of variables, and since $\# \boldsymbol{u} \gneqq k^4$, there is at least one such subset $x_{i_1}, \ldots, x_{i_t}$ with $\#(u_{i_1}, \ldots, u_{i_t}) \gneqq k^2$. The proof now follows from Lemma 2.10. Q.E.D.

§3. The general construction of T_λ

We begin with $\lambda \in H(k^2, k^2; n)$, $\lambda_{k^2} \geqq k^2$, so that $\lambda \mapsto (\mu, \nu')$ as in 0.3. We make the following

3.1. Assumptions.

(a) $\nu_{k^2} \geqq k^4 + k^2$,

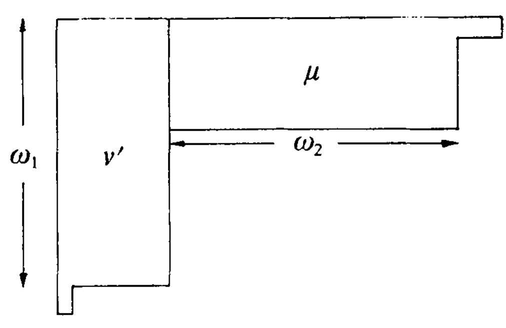
(b) $m_\mu(F_k), m_\nu(F_k) \neq 0$.

3.2. Example. Let $n \geqq 2k^2(k^2 + k^4)$ and choose $n_1 = [n/2]$, $n_2 = n - n_1$: $n_1, n_2 \geqq k^2(k^2 + k^4)$. Let now $n_i = \omega_i k^2 + r_i$ $0 \leqq r_i < k^2$, $i = 1, 2$ (so $\omega_i \geqq k^2 + k^4$) and define

$$\mu = (\omega_2 + r_2, \underbrace{\omega_2, \ldots, \omega_2}_{k^2 - 1}), \quad \nu = (\omega_1 + r_1, \underbrace{\omega_1, \ldots, \omega_1}_{k^2 - 1}).$$

and $\lambda \mapsto (\mu, \nu')$. Thus

$\lambda =$



As was noted in 2.5, $m_\mu(F_k), m_\nu(F_k) \neq 0$.

We now construct T_λ, then show later that $e_{T_\lambda}(x) \notin \mathrm{Id}(E_k)$.

3.3. Constructing T_λ. Recall that $\lambda \vdash n$ $n = n_1 + n_2$, $\nu \vdash n_1$, $\mu \vdash n_2$ $\lambda \to (\mu, \nu')$ and $m_\nu(F_k)$, $m_\mu(F_k) \neq 0$. Thus there are (many) tableaux t_ν (on $1, \ldots, n_1$) such that $f_{t_\nu}(x)$ is not an identity of F_k. We shall pick, in §4, one such tableau T_ν with $f_{T_\nu}(x) \notin \mathrm{Id}(F_k)$.

Similarly, there is a tableau T_μ (on $1, \ldots, n_2$) such that $e_{T\mu}(x)$ is not an identity of F_k.

We construct:

$$T_\lambda = T'_\nu \,\big|\, (T_\mu + n_1).$$

Here T'_ν is the conjugate of T_ν; $T_\mu + n_1$ is the tableau on $n_1 + 1, \ldots, n_1 + n_2$, obtained from T_μ by adding n_1 to each of its entries, and $T'_\nu \,\big|\, (T_\mu + n_1)$ is the "glueing together" of the two tableaux [12, pp. 1422–3].

3.4. REMARKS. Let $S_{n_2}(n_1 + 1, \ldots, n_1 + n_2)$ be the symmetric group on $n_1 + 1, \ldots, n_1 + n_2$ (its order is $n_2!$). Let

$$R_{T_\mu + n_1}, C_{T_\mu + n_1} \subseteq S_{n_2}(n_1 + 1, \ldots, n_1 + n_2)$$

be the row and the column permutations of $T_\mu + n_1$, and define $\bar{R}_{T_\mu + n_1}, \bar{C}_{T_\mu + n_1}$ as in 2.1. Clearly

$$C_{T_\lambda} = C_{T'_\nu} \times C_{T_\mu + n_1}$$

and hence

$$\bar{C}_{T_\lambda} = \bar{C}_{T'_\nu} \cdot \bar{C}_{T_\mu + n_1}.$$

On the other hand,

$$R_{T_\lambda} \supsetneq R_{T'_\nu} R_{T_\mu + n_1}.$$

Choosing a transversal L we obtain

$$R_{T_\lambda} = \bigcup_{\rho \in L} \rho(R_{T'_\nu} \times R_{T_\mu + n_1})$$

a disjoint union. Thus

$$\bar{R}_{T_\lambda} = \sum_{\rho \in L} \rho(\bar{R}_{T'_\nu} \cdot \bar{R}_{T_\mu + n_1}).$$

We choose L such that $1 \in L$. Recall that $n = n_1 + n_2$. With the above notations we prove

3.5. LEMMA. *For some* $0 \neq \beta \in F$,

$$e_{T_\lambda}(x_1, \ldots, x_n) = \beta \sum_{\rho \in L} \rho(e_{T'_\nu}(x_1, \ldots, x_{n_1}) \cdot e_{T_\mu}(x_{n_1+1}, \ldots, x_{n_1+n_2})).$$

PROOF. Note that $e_{T'_\nu} = \alpha_1 \bar{R}_{T'_\nu} \cdot \bar{C}_{T'_\nu}$, $e_{T_\mu} = \alpha_2 \bar{R}_{T_\mu} \cdot \bar{C}_{T_\mu}$, and

$$e_{T_\mu}(x_{n_1+1}, \ldots, x_{n_1+n_2}) = \alpha_2 \bar{R}_{T_\mu + n_1} \cdot \bar{C}_{T_\mu + n_1}.$$

If $p \in R_{T_\mu+n_1}$ and $q \in C_{T'_\nu}$, then

$$pq = qp$$

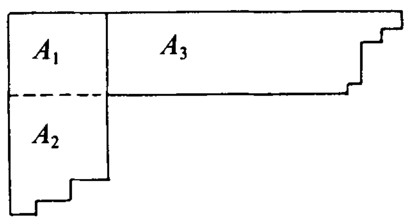
since they permute two disjoint subsets of $\{1, \ldots, n\}$. The proof now follows since

$$e_{T_\lambda} = \alpha \bar{R}_{T_\lambda} \cdot \bar{C}_{T_\lambda} = \alpha \sum_{\rho \in L} \rho(\bar{R}_{T'_\nu} \cdot \bar{R}_{T_\mu+n_1})(\bar{C}_{T'_\nu} \cdot \bar{C}_{T_\mu+n_1})$$

$$= \alpha \sum_{\rho \in L} \rho(\bar{R}_{T'_\nu} \cdot \bar{C}_{T'_\nu} \cdot \bar{R}_{T_\mu+n_1} \cdot \bar{C}_{T_\mu+n_1}). \qquad \text{Q.E.D.}$$

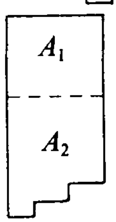
3.6. Notation. Denote the various areas of T_λ as follows:

$T_\lambda =$



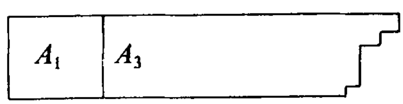
Thus

$T'_\nu =$



and $T_\mu + n_1 \equiv A_3$

3.7. Remark. In 3.4 we can choose the transversal L such that each $\rho \in L$ is a row permutation of the diagram



and satisfying the following property:

If $1 \neq \rho \in L$, then there is an entry i in A_1 (resp. in A_3) such that $\rho(i)$ is an entry of A_3 (resp. of A_1). Also, for any $\rho \in L$, if i is in A_2, then $\rho(i) = i$.

§4. The construction of T'_θ

4.1. Remark. Let T_θ be a tableau of shape θ. Let $\tilde{T}_\theta$ be a tableau which is obtained from T_θ by permuting any set of rows (columns) of equal length. It is easy to show that $R_{\tilde{T}_\theta} = R_{T_\theta}$ and $C_{\tilde{T}_\theta} = C_{T_\theta}$, hence $e_{\tilde{T}_\theta} = e_{T_\theta}$.

4.2. We fix a tableau t_ν for which $f_{t_\nu}(x) \notin \mathrm{Id}(F_k)$ (3.3), then apply the above remark to obtain in 4.6, T_ν. Note that if t'_ν is the conjugate of t_ν, then, by 2.7,

$$e_{t'_\nu}(v_1 \otimes X_1, \ldots, v_{n_1} \otimes X_{n_1}) \neq 0$$

where the v_i's are basis elemlents of V (§1) and the X_i's are generic $k \times k$ matrices.

4.3. Definition. With t_ν as in 4.2, define $S \subseteq \mathbb{N}$ as follows: $s \in S$ if and only if there exist $u_1, \ldots, u_{n_1} \in E$ as in 2.9 with $\#(u_1, \ldots, u_{n_1}) = s$ and such that $e_{t'_\nu}(u_1 \otimes X_1, \ldots, u_{n_1} \otimes X_{n_1}) \neq 0$.

4.4. Remarks.

(a) $0 \in S$, hence $S \neq \varnothing$.

(b) If $s \in S$ then $s \leqq k^4$.

Proof. (a) follows from 4.2, while (b) follows from 2.12, since $t'_\nu(=T_\theta)$ has k^2 columns.

Conclude that there exists $s \in S$ maximal, $0 \leqq s \leqq k^4$. We then *fix* $\boldsymbol{u} = (u_1, \ldots, u_{n_1})$ with $\#\boldsymbol{u} = s$ and

$$e_{t'_\nu}(u_1 \otimes X_1, \ldots, u_{n_1} \otimes X_{n_1}) \neq 0.$$

4.5. Lemma. *Let t_ν and $\boldsymbol{u} = (u_1, \ldots, u_{n_1})$ be as above — with $\#\boldsymbol{u} = s$ maximal in S. Then there exists a tableau T_ν which satisfies*:

(a) $e_{T'_\nu} = e_{t'_\nu}$.

(b) *If u_i is of even length, then i does* not *appear in the first k^2 columns of T_ν (so i does not appear in the first k^2 rows of T'_ν).*

Proof. By 3.1(a), $\nu_{k^2} \geqq k^2 + k^4$. Thus t_ν has $\nu_{k^2} \geqq k^2 + k^4$ columns of height k^2. Given the above $\boldsymbol{u}$, let $1 \leqq i_1, \ldots, i_s \leqq n_1$ be the indices for which the u_{i_j}'s are of even length. These i_j's appear in at most s columns to t_ν, and $s \leqq k^4$. Thus there are at least k^2 columns of height k^2, of t_ν, which do not contain any of these i_j's.

Let T_ν be a tableau which is obtained from t_ν by permuting the columns of t_ν — of height k^2 — in such a way that $i_1, \ldots, i_s$ do *not* appear in the first k^2 columns of T_ν. Such T_ν obviously exists. Then (b) holds by construction, while (a) follows from 4.1. Q.E.D.

4.6. Conclusion. Recall that the tableau T_μ was chosen in 3.3. It is the above tableau T_ν of 4.5 that we choose; then, as mentioned in 3.3, we construct

$$T_\lambda = T'_\nu \,\big|\, (T_\mu + n_1).$$

§5. $e_{T_\lambda}(x) \notin \mathrm{Id}(E_k)$

In order to show that $e_{T_\lambda}(x)$ is not an identity of E_k we construct below a substitution of the form $x_i \to u_i \otimes X_i$, then show that $e_{T_\lambda}(u \otimes X) \neq 0$.

5.1. THE SUBSTITUTION. We choose $u_1, \ldots, u_{n_1}$ as in 4.4, 4.5, then choose $u_{n_1+1} = \cdots = u_{n_1+n_2} = 1$. Now let $X_1, \ldots, X_{n_1+n_2}$ be generic $k \times k$ matrices, and consider the substitution

$$x_i \to u_i \otimes X_i, \qquad 1 \leqq i \leqq n_1 + n_2.$$

With T_λ as in 4.6 and with 4.5 in mind, we now prove

5.2. LEMMA. *Let* $1 \neq \rho \in L$, *L as in* 3.4–3.7, *and let* $x_i \to u_i \otimes X_i$ *as in* 5.1. *Then*

$$\begin{aligned}(\rho[e_{T'_\nu}(x_1, \ldots, x_{n_1})e_{T_\mu}(x_{n_1+1}, \ldots, x_{n_1+n_2})])(x_i \to u_i \otimes X_i)\\ = e_{T'_\nu}(u_{\rho(1)} \otimes X_{\rho(1)}, \ldots, u_{\rho(n_1)} \otimes X_{\rho(n_1)})\\ \times e_{T_\mu}(u_{\rho(n_1+1)} \otimes X_{\rho(n_1+1)}, \ldots, u_{\rho(n_1+n_2)} \otimes X_{\rho(n_1+n_2)})\\ = 0.\end{aligned}$$

PROOF. Consider $1 \leqq i \leqq n_1$. By 3.6, 3.7 and 4.5(b), if u_i has even length, then i appears in A_2, hence $\rho(i) = i$ so $u_{\rho(i)}$ $(= u_i)$ has even length. Also, there exists i_0 in A_1 with $\rho(i_0)$ in $A_3 : n_1 + 1 \leqq \rho(i_0) \leqq n_1 + n_2$. Since $u_{n_1+1} = \cdots = u_{n_1+n_2} = 1$ are of even length, hence so is $u_{\rho(i_0)}$. It follows that

$$\#(u_{\rho(1)}, \ldots, u_{\rho(n_1)}) \geqq s + 1.$$

By 4.3 and the maximality of $s \in S$,

$$e_{T'_\nu}(u_{\rho(1)} \otimes X_{\rho(1)}, \ldots, u_{\rho(n_1)} \otimes X_{\rho(n_1)}) = 0,$$

and the proof follows. Q.E.D.

5.3. REMARK. Let $\{u_i \otimes X_i\}$ as in 5.1. It is easy to see that for an apropriate polynomial $g(x_1, \ldots, x_{n_1})$,

$$0 \neq e_{T'_\nu}(u_1 \otimes X_1, \ldots, u_{n_1} \otimes X_{n_1}) = (u_1 \cdots u_{n_1}) \otimes g(X_1, \ldots, X_{n_1}).$$

5.4. THEOREM. *With* $x_i \to u_i \otimes X_i$ *as in* 5.1 *and* $g(x)$ *as in* 5.3 *we have*:

$$\begin{aligned}e_{T_\lambda}(u \otimes X) &= e_{T_\lambda}(u_1 \otimes X_1, \ldots, u_{n_1+n_2} \otimes X_{n_1+n_2})\\ &= (u_1 \cdots u_{n_1}) \otimes [(g(X_1, \ldots, X_{n_1}) \cdot e_{T_\mu}(X_{n_1+1}, \ldots, X_{n_1+n_2})].\end{aligned}$$

In particular, $e_{T_\lambda}(u \otimes X) \neq 0$.

PROOF. By 3.5 and 5.2 and 5.3,

$$\begin{aligned}\beta^{-1} e_{T_\lambda}(u \otimes X) &= e_{T'_\nu}(u_1 \otimes X_1, \dots, u_{n_1} \otimes X_{n_1}) \cdot e_{T_\mu}(X_{n_1+1}, \dots, X_{n_1+n_2}) \\ &\quad + \sum_{1 \neq \rho \in L} e_{T'_\nu}(u_{\rho(1)} \otimes X_{\rho(1)}, \dots, u_{\rho(n_1)} \otimes X_{\rho(n_1)}) \\ &\quad \times e_{T_\mu}(u_{\rho(n_1+1)} \otimes X_{\rho(n_1+1)}, \dots, u_{\rho(n_1+n_2)} \otimes X_{\rho(n_1+n_2)}) \\ &= e_{T'_\nu}(u_1 \otimes X_1, \dots, u_{n_1} \otimes X_{n_1}) \cdot e_{T_\mu}(X_{n_1+1}, \dots, X_{n_1+n_2}) \\ &= (u_1 \cdots u_{n_1}) \otimes (g(X_1, \dots, X_{n_1}) e_{T_\mu}(X_{n_1+1}, \dots, X_{n_1+n_2}).\end{aligned}$$

Now, $g(X_1, \dots, X_{n_1}) \neq 0$ by 5.3, and $e_{T_\mu}(X_{n_1+1}, \dots, X_{n_1+n_2}) \neq 0$ since $e_{T_\mu} \notin \mathrm{Id}(E_k)$ and the X_i's are generic. By Amitsur's primeness theorem $g(X_1, \dots, X_{n_1}) \cdot e_{T_\mu}(X_{n_1+1}, \dots, X_{n_1+n_2}) \neq 0$, and hence $e_{T_\lambda}(u \otimes X) \neq 0$. Q.E.D.

5.5. REMARKS AND CONJECTURES. Let $\lambda \mapsto (\mu, \nu')$ as in 5.4. It follows from 5.4 that $m_\lambda(F_k) \geqq 1$. Recall that T_λ was constructed (3.3) from t_ν and T_μ. In fact, there are $m_\nu(F_k)$ tableaux $\{t_\nu\}$ with $\{e_{t_\nu}(x)\}$ independent over FS_{n_1} and modulo $\mathrm{Id}(F_k)$. Likewise we could have chosen $m_\mu(F_k)$ tableaux $\{T_\mu\}$, etc.

We could have therefore constructed $m_\nu(F_k) \cdot m_\mu(F_k)$ corresponding tableaux $\{T_\lambda\}$, and we conjecture that $\{e_{T_\lambda}(x)\}$ are independent over FS_n and modulo $\mathrm{Id}(E_k)$. In other words, we have

CONJECTURE. Let $\lambda \mapsto (\mu, \nu')$ as above, then

$$m_\lambda(E_k) \geqq m_\mu(F_k) \cdot m_\nu(F_k).$$

We also guess — but dare not conjecture — that

$$m_\lambda(E_k) \approx m_\mu(F_k) \cdot m_\nu(F_k).$$

§6. Applications: bounds for $c_n(E_k)$

In the next two sections we apply Theorem 5.4, together with some other results, to give the bounds for the codimensions that were promised in Theorem 0.1.

We begin with $c_n(E_k)$. The upper bound follows easily:

6.1. LEMMA. *There are constants c_2 and g_2 such that for all n*,

$$c_n(E_k) \leqq c_2 \cdot \left(\frac{1}{n}\right)^{g_2} \cdot (2 \cdot k^2)^n.$$

PROOF. Note that $E_k = F_k \otimes E$. By [15],

$$c_n(F_k) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot \left(\frac{1}{2}\right)^{(k^2-1)/2} \cdot 1! \cdots (k-1)! \cdot k^{(k^2+4)/2} \left(\frac{1}{n}\right)^{(k^2-1)/2} \cdot k^{2n}$$

and by [8], $c_n(E) = 2^{n-1}$. By [10],

$$c_n(E_k) = c_n(F_k \otimes E) \leqq c_n(F_k) \cdot c_n(E),$$

and the proof follows. Q.E.D.

Note that the proof gives c_2 and g_2 explicitly!

6.2. THEOREM. *There are constants c_1, c_2, g_1 and g_2 such that for all n,*

$$c_1 \left(\frac{1}{n}\right)^{g_1} (2 \cdot k^2)^n \leqq c_n(E_k) \leqq c_2 \left(\frac{1}{n}\right)^{g_2} (2 \cdot k^2)^n.$$

PROOF. The upper bound is given by 6.1. It suffices to prove the lower bound for n large enough. Let n be large, in particular let $n \geqq 2k^2(k^2 + k^4)$ and let $\lambda \mapsto (\mu, \nu')$ be as in Example 3.2. Thus $m_\nu(F_k)$, $m_\mu(F_k) \neq 0$, hence by Theorem 5.4, which was proved under such assumptions, $m_\lambda(E_k) \neq 0$. Hence

$$c_n(E_k) \geqq d_\lambda.$$

We shall complete the proof by estimating d_λ asymptotically. The main tool here is [2, §7] (in particular, 7.14.1 there).

Let $\bar{\nu}$ be the diagram obtained from ν by removing the first $k^2 \times k^2$ rectangle: $\bar{\nu} = (\omega_1 - k^2 + r_1, (\omega_1 - k^2)^{k^2-1})$. By [2, 7.14.1],

$$d_\lambda = \frac{n!}{(n-k^4)!} \cdot \binom{n-k^4}{n_2} \cdot d_{\bar{\nu}} \cdot d_\mu \cdot \left(\prod_{(i,j)\in R} h_{ij}\right)^{-1} \tag{6.2.1}$$

(h_{ij} are the "hook" numbers, and R is that $k^2 \times k^2$ corner rectangle).

Now

$$\frac{n!}{(n-k^4)!} \simeq n^{k^4},$$

while for $(i,j) \in R$, $h_{i,j} \approx n/k^2$ so that

$$\left(\prod_{(i,j)\in R} h_{ij}\right)^{-1} \approx \left(\frac{k^2}{n}\right)^{k^4}.$$

Thus

(6.2.2) $$\frac{n!}{(n-k^4)!}\left(\prod_R h_{ij}\right)^{-1} \approx k^{2k^4}$$

which is a constant.

Since $n_2 \approx (n-k^4)/2$ and n is large, hence

(6.2.3) $$\binom{n-k^4}{n_2} \simeq c \cdot \frac{1}{\sqrt{n}} 2^n$$

for some constant c.

We now estimate d_μ (and similarly d_φ): Recall that $\mu = (\omega_2 + r_2, \omega_2^{k^2-1})$. Let

$$D(x_1, \ldots, x_m) = \prod_{1 \leqq i < j \leqq m} (x_i - x_j).$$

By the Young–Frobenius formula $d_\mu = d_1 \cdot d_2$ where

$$d_1 = \frac{n_2!}{\omega_2!(\omega_2+1)!\cdots(\omega_2+k^2-2)!(\omega_2+r_2+k^2-1)!}$$

and

$$d_2 = D(\omega_2, \omega_2+1, \ldots, \omega_2+k^2-2, \omega_2+r_2+k^2-1).$$

Now, $D(\omega_2, \ldots, \omega_2+k^2-2, \omega_2+r_2+k^2-1) = D(1, 2, \ldots, k^2-1, r_2+k^2)$ is a polynomial in r_2; since $0 \leqq r_2 \leqq k^2-1$, that polynomial is bounded.

To estimate d_1 (of d_μ), apply Stirling's formula: if n is large and a is bounded, then

$$(n+a)! \simeq \sqrt{2\pi} e^{-n} n^{n+a} \sqrt{n}.$$

Since ω_2 is large, for all $0 \leqq j \leqq r_2 + k^2 - 1$, $(\omega_2+j)^{\omega_2+j} \simeq e^j \cdot \omega_2^{\omega_2+j}$, and $\sqrt{\omega_2+j} \simeq \sqrt{\omega_2}$. Also note that

$$\omega_2 + (\omega_2+1) + \cdots + (\omega_2+k^2-2) + (\omega_2+r_2+k^2-1) = n_2 + h \qquad \text{where}$$

$$h = \tfrac{1}{2}k(k-1).$$

Then

$$d_1 \simeq \bar{c} \cdot \frac{n_2^{n_2}\sqrt{n_2}}{\omega_2^{n_2+h} \cdot \sqrt{\omega_2}^{k^2}} \qquad (\bar{c} \text{ is a constant})$$

$$= \bar{c} \cdot \left(\frac{n_2}{\omega_2}\right)^{n_2} \cdot \frac{\sqrt{n_2}}{\omega_2^{h+k^2/2}}.$$

Now, $n_2/\omega_2 = k^2(1 + r_2/k^2\omega_2)$ and $k^2\omega_2 = n_2 - r_2$, hence

$$\left(\frac{n_2}{\omega_2}\right)^{n_2} = k^{2n_2}\cdot\left(1+\frac{r_2}{k^2\omega_2}\right)^{n_2-r_2}\left(1+\frac{r_2}{k^2\omega_2}\right)^{r_2} \simeq e^{r_2}\cdot k^{2n_2}.$$

Since $\omega_2 \simeq n_2/k^2$ and $h + \frac{1}{2}k^2$ is bounded, it follows that

$$d_1 \simeq c'\cdot\left(\frac{1}{n_2}\right)^{g'}\cdot k^{2n_2}, \qquad c', g' \text{ constants.}$$

Assume for simplicity, that n is even. Thus $2n_2 = n$ so $k^{2n_2} = k^n$ and we have that

$$d_\mu \simeq \bar{c}\left(\frac{1}{n}\right)^{\bar{g}} k^n. \tag{6.2.4}$$

Similarly,

$$d_{\bar{\nu}} \simeq \tilde{c}\left(\frac{1}{n}\right)^{\tilde{g}} k^n. \tag{6.2.5}$$

Here $\bar{c}$, $\tilde{c}$, $\bar{g}$, $\tilde{g}$ are (explicit) constants.

Combining (6.2.1)–(6.2.5) we obtain that

$$d_\lambda \simeq c_1\cdot\left(\frac{1}{n}\right)^{g_1}(2k^2)^n, \qquad c_1, g_1 \text{ (explicit) constants.}$$

Since $d_\lambda < c_n(E_k)$, the proof follows. Similarly when n is odd. Q.E.D.

§7. The algebras $F_{k,l}$

In this section we prove Theorem 0.1 for the algebras $E_{k,l}$ of §1. The upper bound follows from the following three theorems:

7.1. THEOREM (Berele [1]). *We have*

$$\chi_n(E_{k,l}) = \sum_{\lambda\in H(k^2+l^2,2kl;n)} m_\lambda(E_{k,l})\cdot\chi_\lambda$$

(*and the two indices* $k^2 + l^2$ *and* $2kl$ *are minimal*).

7.2. THEOREM (Berele, Regev [3]). *Let A be any P.I. algebra, $\chi_n(A) = \Sigma_{\lambda\in\mathrm{Pan}(n)} m_\lambda(A)\cdot\chi_\lambda$ its cocharacters. Then there exists an r such that for all n and for all $\lambda\vdash n$, $m_\lambda(A) \leqq n^r$.*

7.3. THEOREM (Berele, Regev [2, Th.7.21]).

$$\sum_{\lambda \in H(k,l;n)} d_\lambda \underset{n\to\infty}{\simeq} c \cdot \left(\frac{1}{n}\right)^g \cdot (k+l)^n$$

where c, g *are (explicit) constants.*

As a corollary we have

7.4. LEMMA. *There are (explicit) constants* c_2, g_2 *such that*

$$\chi_n(E_{k,l}) \leqq c_2 \left(\frac{1}{n}\right)^{g_2} \cdot (k+l)^{2n}.$$

PROOF. By 7.1, 7.2 and 7.3,

$$c_n(E_{k,l}) \leqq n^r \sum_{\lambda \in H(k^2+l^2, 2kl; n)} d_\lambda \simeq n^r \cdot c \cdot \left(\frac{1}{n}\right)^g \cdot ((k^2+l^2)+(2kl))^n,$$

and the proof follows. Q.E.D.

We are now ready to prove

7.5. THEOREM. *There exist (explicit) constants* c_1, c_2, g_1, g_2 *such that*

$$c_1 \cdot \left(\frac{1}{n}\right)^{g_1} \cdot (k+l)^{2n} \leqq c_n(E_{k,l}) \leqq c_2 \left(\frac{1}{n}\right)^{g_2} \cdot (k+l)^{2n}.$$

PROOF. Lemma 7.4 gives the upper bound. To obtain the lower bound, recall that $E_{k,l} \otimes_F E \sim E_{k+l}$ (Theorem 1.2), hence

$$c_n(E_{k+l}) = c_n(E_{k,l} \otimes E) \leqq c_n(E_{k,l}) \cdot c_n(E) = c_n(E_{k,l}) \cdot 2^{n-1}.$$

Thus

$$c_n(E_{k,l}) \geqq \frac{1}{2^{n-1}} c_n(E_{k+l}),$$

and the proof follows from Theorem 6.2. Q.E.D.

With Theorem 0.3, 6.2 and 7.5 in mind, we make the following

7.6. CONJECTURE. Let $A = E_k$ or $E_{k,l}$. Then there are constants c, g, a such that

$$c_n(A) \underset{n\to\infty}{\simeq} c \cdot \left(\frac{1}{n}\right)^g \cdot a^n.$$

In other words we conjecture that a property similar to 0.3 holds for any K-prime algebra.

REFERENCES

1. A. Berele, *Cocharacters of* $\mathbf{Z}/2$ $\mathbf{Z}$*-graded algebras*, preprint.

2. A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Advances in Math. **64** (1987), 118–175.

3. A. Berele and A. Regev, *Applications of hook Young diagrams to P.I. algebras*, J. Algebra **82** (1983), 559–567.

4. E. Formanek, *Invariants of the ring of generic matrices*. J. Algebra **89** (1984), 178–223.

5. E. Formanek, *A conjecture of Regev on the Capelli polynomial*, J. Algebra, to appear.

6. T. V. Gateva, *On the P.I. degree of the tensor product of P.I.-algebras*, Comptes Rendus Acad. Bulg. Sci. **36**(11)(1983).

7. A. R. Kemer, Summary of XVI All-Union Algebraic Conference, Leningrad, 1981 (Russian).

8. D. Krakowski and A. Regev, *The polynomial identities of the Grassmann algebra*, Trans. Am. Math. Soc. **181** (1973), 429–438.

9. J. B. Olson and A. Regev, *Colength sequences of some T-ideals*, j. Algebra **38** (1976), 100–111.

10. A. Regev, *Existence of identities in* $A \otimes B$, Isr. J. Math. **11** (1972), 131–152.

11. A. Regev, *Algebras satisfying a Capelli identity*, Isr. J. Math. **33** (1979), 149–154.

12. A. Regev, *The polynomial identities of matrices in characteristic zero*. Commun. Algebra **8** (15) (1980), 1417–1467.

13. A. Regev, *The Kronecker product of* S_n*-characters and an* $A \otimes B$ *theorem for Capelli identities*, J. Algebra **66** (1980), 505–510.

14. A. Regev, *A polynomial rate of growth for the multiplicities in cocharacters of matrices*, Isr. J. Math. **42** (1982), 65–74.

15. A. Regev, *Codimensions and trace codimensions of matrices are asymptotically equal*, Isr. J. Math. **47** (1984), 246–250.